A Structural Closed-Form Representation of the Odd Collatz Iteration

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Abstract

We derive an exact closed-form expression for the k-step iteration of the odd Collatz map

$$T(x) = \frac{3x+1}{2^{v_2(3x+1)}},$$

and obtain a structural cycle condition

$$(2^A - 3^k)x_0 = S,$$

where A and S encode, respectively, the cumulative 2-adic reductions and the weighted contributions of all additive increments. This decomposition separates the multiplicative, additive, and 2-adic components of the dynamics and provides a representation independent of binary expansion.

1 Introduction

Consider the odd Collatz map

$$x_{i+1} = \frac{3x_i + 1}{2^{a_i}}, \qquad a_i = v_2(3x_i + 1),$$

where all x_i remain odd. Although the iteration is elementary, its structure becomes transparent only after separating the multiplicative effect of repeated factors of 3 from the additive effect of the "+1" and the 2-adic reductions. We derive an exact closed-form expression for x_k and obtain a structural cycle condition.

2 Closed-form expression for x_k

Theorem 1. *After k iterations of the odd Collatz map,*

$$x_k = \frac{3^k x_0 + S}{2^A},$$

where

$$A = \sum_{i=0}^{k-1} a_i, \qquad S = \sum_{j=0}^{k-1} 3^{k-1-j}\, 2^{A_j}, \qquad A_j = \sum_{i=0}^{j-1} a_i.$$

Proof. The base case $k = 1$ gives

$$x_1 = \frac{3x_0 + 1}{2^{a_0}},$$

so $A = a_0$ and $S = 1$.

Assume the formula holds for k. Then

$$x_{k+1} = \frac{3x_k + 1}{2^{a_k}} = \frac{3(3^k x_0 + S_k)/2^{A_k} + 1}{2^{a_k}} = \frac{3^{k+1}x_0 + 3S_k + 2^{A_k}}{2^{A_k + a_k}}.$$

Thus

$$A_{k+1} = A_k + a_k, \qquad S_{k+1} = 3S_k + 2^{A_k},$$

and unrolling this recurrence yields the stated closed form. □

3 Structure of the additive term S

The term S collects all contributions arising from the "+1" in the map. Its structure becomes transparent once we track how each such increment propagates through the iteration.

At step j, the expression $3x_j + 1$ contributes a single additive unit. After this moment, it is multiplied by 3 at every subsequent step, and thus accumulates exactly $k-1-j$ factors of 3. Before step j, the trajectory has already undergone j divisions by powers of 2, contributing the factor 2^{A_j}.

Therefore the contribution of step j is

$$3^{k-1-j} \cdot 2^{A_j},$$

and summing over all steps gives

$$S = \sum_{j=0}^{k-1} 3^{k-1-j}\, 2^{A_j}.$$

The geometric component can be written in either direction:

$$\sum_{j=0}^{k-1} 3^{k-1-j} = \sum_{j=0}^{k-1} 3^j,$$

since the two sums list the same powers of 3 in reverse order. Thus S is a weighted geometric progression, where the powers of 3 reflect how long each "+1" travels forward through the iteration, and the powers of 2 encode the cumulative 2-adic reductions that occurred before that step.

4 Cycle condition

A cycle of length k satisfies $x_k = x_0$. Substituting the closed form gives

$$x_0 = \frac{3^k x_0 + S}{2^A},$$

so

$$(2^A - 3^k)x_0 = S.$$

For normalized $x_0 = 1$ this becomes

$$3^k + S = 2^A.$$

5 Example: $k = 3$

Let $x_0 = 7$. The iteration proceeds as follows:

$$3 \cdot 7 + 1 = 22 = 2 \cdot 11, \quad a_0 = 1, \quad x_1 = 11,$$

$$3 \cdot 11 + 1 = 34 = 2 \cdot 17, \quad a_1 = 1, \quad x_2 = 17,$$

$$3 \cdot 17 + 1 = 52 = 4 \cdot 13, \quad a_2 = 2, \quad x_3 = 13.$$

Thus $A = 1 + 1 + 2 = 4$ and

$$S = 3^2 + 3^1 2^1 + 3^0 2^2 = 9 + 6 + 4 = 19.$$

The closed form gives

$$x_3 = \frac{3^3 \cdot 7 + 19}{2^4} = 13.$$

6 Impossibility of a 2-cycle

Assume a 2-cycle with $x_0 = 1$. Then

$$3^2 + S = 2^A.$$

The left-hand side is odd, since S is odd (it contains exactly one odd term and all others are even). The right-hand side is even. Thus no equality is possible, and no 2-cycle exists.

7 Example: $k = 4$ with mixed exponents

Let $x_0 = 5$. The iteration is

$$3 \cdot 5 + 1 = 16 = 2^4 \cdot 1, \quad a_0 = 4, \quad x_1 = 1,$$

$$3 \cdot 1 + 1 = 4 = 2^2 \cdot 1, \quad a_1 = 2, \quad x_2 = 1,$$

$$3 \cdot 1 + 1 = 4 = 2^2 \cdot 1, \quad a_2 = 2, \quad x_3 = 1,$$

$$3 \cdot 1 + 1 = 4 = 2^2 \cdot 1, \quad a_3 = 2, \quad x_4 = 1.$$

Thus $A = 4 + 2 + 2 + 2 = 10$ and

$$S = 27 + 144 + 192 + 256 = 619.$$

The closed form gives

$$x_4 = \frac{3^4 \cdot 5 + 619}{2^{10}} = 1.$$

8 Conclusion

We derived an exact algebraic representation of the odd Collatz iteration and a structural cycle condition. The decomposition

$$x_k = \frac{3^k x_0 + S}{2^A}$$

separates the multiplicative, additive, and 2-adic components of the dynamics and appears not to be present in existing literature.

References

[1] J. C. Lagarias, *The* $3x + 1$ *problem and its generalizations*, American Mathematical Monthly, 92(1), 1985.